Solutions to Question Sheet 4, Continuity I v1 2019-20

Continuous functions

Six examples of showing a function is continuous.

1. Let
$$f(x)=\frac{x^2-2x-15}{x+3}, x\neq -3.$$
How should $f(-3)$ be defined so that f is continuous at -3?

Solution Recall the definition that f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$. Since
$$\frac{x^2-2x-15}{x+3}=\frac{(x+3)(x-5)}{x+3}=x-5$$
for all $x\neq -3$, we have
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}=\lim_{x\to -3}(x-5)=-8.$$
Thus choose $f(-3)=-8$.

2. Prove, by verifying the ε - δ definition that $h(x)=|x|$ is continuous at $x=0$.

Deduce that h is continuous on $\mathbb{R}$.

(You need not verify the definition for $x\neq 0$, instead quote results from the lecture notes.)

Solution We first prove that h is continuous at 0 by verifying the ε - δ definition. Let $\varepsilon>0$ be given, choose $\delta=\varepsilon$. and assume $|x-0|<\delta$. Then
$$|h(x)-0|=||x|-0|=|x|<\delta=\varepsilon$$
as required.

Note If you had not been asked to verify the ε - δ definition you could have examined the two one-sided limits

$$\begin{aligned}\lim_{x\to 0+} h(x) &= \lim_{x\to 0+} |x| = \lim_{x\to 0+} x = 0,\\ \lim_{x\to 0-} h(x) &= \lim_{x\to 0-} |x| = \lim_{x\to 0+} (-x) = 0.\end{aligned}$$

Since both limits exist and are equal we can say $\lim_{x\to 0} h(x) = 0$. And since $0 = h(0)$ we have $\lim_{x\to 0} h(x) = h(0)$, which is the definition that h is continuous at $x = 0$. **End of Note**

To show that $|x|$ is continuous on **all** of $\mathbb{R}$ it remains to prove it is continuous at all $x \neq 0$. The question explicitly says that you are **not** required to verify the ε - δ definition for such x. Instead we quote results from the course.

If $x > 0$ then $h(x) = |x| = x$, a polynomial of x, so $h(x)$ is continuous.

If $x < 0$ then $h(x) = |x| = -x$, a polynomial of x, so $h(x)$ is again continuous.

Thus h is continuous for all $x \in \mathbb{R}$, i.e. it is continuous on $\mathbb{R}$.

3. Prove, by verifying the ε - δ definition that

   i) the function $f(x) = x^2$ is continuous on $\mathbb{R}$,

      Hint Look back at Question 2 on Question Sheet 1 and replace $a = 2$ seen there by any $a \in \mathbb{R}$.

   ii) the function $g(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

      Hint Look back at Question 11 on Question Sheet 1 and replace the $a = 9$ seen there by any $a > 0$.

   iii) the function

   $$h(x) = \begin{cases} x^2 + x & \text{for } x \leq 1 \\ \sqrt{x+3} & \text{for } x > 1, \end{cases}$$

      is continuous at $x = 1$.

      Hint Verify the ε - δ definitions of both one-sided limits separately at $x = 1$.

iv) the function

$$\frac{1}{x^2+1}$$

is continuous on $\mathbb{R}$.

Solution i) *Rough Work.* Let $a \in \mathbb{R}$ be given. Assume $|x-a| < \delta$ (remember, that when looking at continuity we do **not** have to exclude $x = a$). Consider

$$|f(x) - f(a)| = \left|x^2 - a^2\right| = |x-a|\,|x+a| < \delta\,|x+a|\,.$$

Recall the idea that if x is 'close' to a then $|x+a|$ should be 'close' to $2\,|a|$, in particular $|x+a|$ will not be much larger than $2\,|a|$. A way of implementing this idea is to assume $|x-a|$ is small and use this to estimate $|x+a|$ by rewriting this so we see $x-a$, i.e. as

$$|x+a| = |(x-a)+2a|\,.$$

In detail, assume $\delta \le 1$ in which case $|x-a| \le 1$. Then

$$\begin{aligned} |x+a| &= |(x-a)+2a| \\ &\le |x-a| + 2\,|a| \quad \text{by triangle inequality} \\ &\le 1 + 2\,|a|\,. \end{aligned}$$

(This is where we see that $|x+a|$ is not be much larger than $2\,|a|$.) Thus $|f(x) - f(a)| < \delta\,(1+2\,|a|)$ which we can ensure is $< \varepsilon$ if we demand $\delta \le \varepsilon/(1+2\,|a|)$.

End of Rough Work.

Note the most commonly seen **error** here is the following:

$$\begin{aligned} 0 < |x-a| < \delta \le 1 \quad &\Longrightarrow \quad -1 < x-a < 1 \\ &\Longrightarrow \quad 2a-1 < x+a < 2a+1 \\ &\Longrightarrow \quad |x+a| < |2a+1|\,. \end{aligned}$$

Yet this is **wrong**. What would this be saying if $a = -1/2$? What is wrong with this sequence of implications? **End of Note**

Solution Let $a \in \mathbb{R}$ and $\varepsilon > 0$ be given. Choose

$$\delta = \min\left(1, \frac{\varepsilon}{1+2|a|}\right).$$

Assume $|x - 0| < \delta$. Then

$$\begin{aligned}
|f(x) - f(a)| &= |x-a|\,|x+a| \\
&= |x-a|\,|(x-a)+2a| \\
&\leq |x-a|\,(|x-a| + 2|a|) && \text{by triangle inequality,} \\
&< \delta\,(1+2|a|) && \text{since } |x-a| < \delta \leq 1 \\
&< \left(\frac{\varepsilon}{1+2|a|}\right)(1+2|a|) && \text{since } \delta \leq \varepsilon/(1+2|a|) \\
&= \varepsilon.
\end{aligned}$$

Hence we have verified the ε - δ definition that f is continuous at a.

True for all $a \in \mathbb{R}$ means that f is continuous on $\mathbb{R}$.

ii) If you look back at Question 11 on Sheet 1 you see that to verify the ε - δ definition of $\lim_{x \to 9} \sqrt{x} = 3$ we required $\delta \leq 9$. When replacing 9 by any $a > 0$ we look at x satisfying $|x - a| < \delta$, i.e. $x \in (a - \delta, a + \delta)$. If $a - \delta < 0$ then the interval $(a - \delta, a + \delta)$ will contain negative x yet for $\sqrt{x}$ to be defined we require $x \geq 0$. Hence we require $a - \delta \geq 0$, i.e. $\delta \leq a$.

Let $a > 0$ and $\varepsilon > 0$ be given. Choose $\delta = \min(a, \varepsilon\sqrt{a})$. Assume $0 < |x - a| < \delta$.

Then $-\delta < x - a < \delta$. Since $\delta \leq a$ the lower bound becomes $-a < x - a$, i.e. $x > 0$ and thus $g(x) = \sqrt{x}$ is well-defined.

We start with a "trick" seen in Sheet 1, based on the difference of squares,

$$\begin{aligned}
|g(x) - g(a)| &= \left|\sqrt{x} - \sqrt{a}\right| = \left|\left(\sqrt{x} - \sqrt{a}\right)\frac{(\sqrt{x} + \sqrt{a})}{(\sqrt{x} + \sqrt{a})}\right| \\
&= \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{|x-a|}{\sqrt{a}} \qquad \text{having used } \sqrt{x} > 0, \\
&< \frac{\delta}{\sqrt{a}} \le \frac{\varepsilon\sqrt{a}}{\sqrt{a}} \qquad \text{since } \delta \le \varepsilon\sqrt{a}, \\
&= \varepsilon.
\end{aligned}$$

Hence we have verified the ε - δ definition of

$$\lim_{x\to a} g(x) = g(a)\,,$$

i.e. that g is continuous at a.

True for all $a > 0$ means that g is continuous on $(0, \infty)$.

Note A not uncommon error was to misinterpret the hint and start with

$$\left|\sqrt{x} - \sqrt{a}\right| = \left|x^{1/4} - x^{1/4}\right| \left|x^{1/4} + x^{1/4}\right|.$$

Unfortunately this makes everything more complicated rather than simpler. **End of Note**

iii) Because f is given by different formula for $x > 1$ and $x < 1$ we need to examine the two one-sided limits and show that

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1) = 2.$$

Let $\varepsilon > 0$ be given.

For the limit from below, i.e. as $x \to 1-$. Choose $\delta = \min(1, \varepsilon/3)$. Assume $1 - \delta < x < 1$.

Then $\delta \le 1$ implies $0 < x < 1$ and thus $|x + 2| < 3$. Therefore

$$\begin{aligned}
|f(x) - 2| &= \left|x^2 + x - 2\right| = |(x+2)(x-1)| \\
&\le 3|x-1| \le 3\delta \le 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.
\end{aligned}$$

Thus we have verified the ε - δ definition of the one-sided limit

$$\lim_{x\to 1-} f(x) = 2.$$

For the limit from above, i.e. as $x \to 1+$. Choose $\delta = \varepsilon$. Assume $1 < x < 1 + \delta$, which will be used below as $x - 1 < \delta$.

Then using a “trick” seen in the solution to the previous question,

$$\begin{aligned} |f(x) - 2| &= \sqrt{x+3} - 2 = \left(\sqrt{x+3} - 2\right) \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \\ &= \frac{(x+3)-4}{\sqrt{x+3}+2} = \frac{x-1}{\sqrt{x+3}+2} \\ &\leq x - 1, \end{aligned} \tag{1}$$

using $\sqrt{x+3} + 2 \geq 1$ (and $x - 1$ positive). Hence

$$|f(x) - 2| \leq x - 1 < \delta = \varepsilon.$$

Thus we have verified the ε - δ definition of the one-sided limit

$$\lim_{x\to 1+} f(x) = 2.$$

Note It would be reasonable, since $x > 1$, to say $\sqrt{x+3} + 2 > 4$ and thus

$$\frac{x-1}{\sqrt{x+3}+2} < \frac{x-1}{4} < \frac{\delta}{4}.$$

You would then choose $\delta = 4\varepsilon$. **End of Note**

iv) Let $a \in \mathbb{R}$ be given.

Rough Work.

Consider

$$\left|\frac{1}{1+x^2}-\frac{1}{1+a^2}\right| = \left|\frac{a^2-x^2}{(1+x^2)(1+a^2)}\right| \le \left|x^2-a^2\right|,$$

having used $1+x^2 \ge 1$, $1+a^2 \ge 1$ and $|a^2-x^2| = |x^2-a^2|$. But now we are back in part (i) where we are trying to show that $|x^2-a^2| < \varepsilon$. Thus choose δ as we did there.

End of Rough work

Solution Let $a \in \mathbb{R}$ and $\varepsilon > 0$ be given. Choose

$$\delta = \min\left(1, \frac{\varepsilon}{1+2|a|}\right).$$

Assume $0 < |x-a| < \delta$. Then, starting as in the rough work,

$$\begin{aligned}
\left|\frac{1}{1+x^2}-\frac{1}{1+a^2}\right| &\le \left|x^2-a^2\right| = |x-a|\,|x+a| \\
&= |x-a|\,|(x-a)+2a| \\
&\le |x-a|\,(|x-a|+2|a|) && \text{by triangle inequality,} \\
&< |x-a|\,(1+2|a|) && \text{since } |x-a| < \delta \le 1 \\
&< \left(\frac{\varepsilon}{1+2|a|}\right)(1+2|a|) \\
&\qquad \text{since } |x-a| < \delta \le \varepsilon/(1+2|a|) \\
&= \varepsilon.
\end{aligned}$$

Hence we have verified the ε - δ definition that $1/(1+x^2)$ is continuous at a.

True for all $a \in \mathbb{R}$ means that $1/(1+x^2)$ is continuous on $\mathbb{R}$.

4. Are the following functions continuous on the domains given or not?

Either prove that they are continuous *by using the appropriate Continuity Rules*, or show they are not.

i)
$$f(x) = \frac{x+2}{x^2+1} \text{ on } \mathbb{R}.$$

ii)
$$g(x) = \frac{3+2x}{x^2-1},$$

firstly on $[-1/2, 1/2]$, secondly on $[-2, 2]$.

iii)
$$h(x) = \frac{x^2+x-2}{(x^2+1)(x-1)} \text{ on } \mathbb{R}.$$

iv)
$$j(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x-2 & \text{if } x > 1. \end{cases}.$$

v)
$$k(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}.$$

vi)
$$\ell(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & x \neq 0 \\ 1 & x = 0. \end{cases}.$$

Solution i) The given function f is a quotient of polynomials, i.e. a rational function. The polynomials are continuous everywhere. Hence f is continuous wherever it is defined. The denominator, x^2+1, is never zero for $x \in \mathbb{R}$, so f is defined everywhere. Hence f is continuous everywhere.

ii) The argument is as in part i). But now the denominator is x^2-1 which is zero at $x = \pm 1$. So

- g is well-defined throughout $[-1/2, 1/2]$ and so g is continuous on $[-1/2, 1/2]$, but

- g is not defined everywhere in $[-2,2]$ and, in fact, g is continuous on $[-2,2]$ except at -1 and 1.

iii) As written, h is defined everywhere except at $x = 1$. So h is continuous on $\mathbb{R}\backslash\{1\}$.

Note When $x = 1$ the numerator is also 0. In fact $x^2 + x - 2 = (x+2)(x-1)$ and thus

$$h(x) = \frac{(x+2)(x-1)}{(x^2+1)(x-1)} = \frac{x+2}{x^2+1}.$$

In this way we could *extend* the definition of h to all of $\mathbb{R}$ but we would then have a *different* function.

iv) $j(x)$ is continuous on $\mathbb{R}$ except possibly at $x = -1$ and $x = 1$.

At $x = -1$ the two one-sided limits are

$$\begin{aligned} \lim_{x\to -1-} j(x) &= \lim_{x\to -1-} (x+2) = 1, \\ \lim_{x\to -1+} j(x) &= \lim_{x\to -1+} x^2 = 1. \end{aligned}$$

Since the two one-sided limits exist and are equal we deduce that $\lim_{x\to -1} j(x) = 1$. Yet $1 = j(-1)$ so $\lim_{x\to -1} j(x) = j(-1)$ which is the definition that j is continuous at $x = -1$.

At $x = 1$ the two one-sided limits are

$$\begin{aligned} \lim_{x\to 1-} j(x) &= \lim_{x\to 1-} x^2 = 1, \\ \lim_{x\to 1+} j(x) &= \lim_{x\to 1+} (x-2) = -1. \end{aligned}$$

Different one-sided limits means that $\lim_{x\to 1} j(x)$ does **not** exist and so cannot equal $j(1)$. Thus j is **not** continuous at $x = 1$.

v) If $x \neq 0$ then $k(x) = (\sin x)/x$. We have shown that $\sin x$ is continuous, as is x, for $x \neq 0$. Hence $k(x)$ is continuous for $x \neq 0$ by the Quotient Rule.

If $x = 0$ we have

$$\lim_{x\to 0} k(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

a result seen in the lectures. By definition, $k(0) = 1$, thus $\lim_{x\to 0} k(x) = k(0)$ and so k is continuous at $x = 0$.

Hence k is continuous on $\mathbb{R}$.

vi) If $x \neq 0$ then $\ell(x) = (1 - \cos x)/x^2$. We have shown that $\cos x$ is continuous, as is x^2, for $x \neq 0$. Hence $\ell(x)$ is continuous for $x \neq 0$ by the Quotient Rule.

If $x = 0$ we have

$$\lim_{x\to 0} \ell(x) = \lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2},$$

a result seen in the lectures. By definition, $\ell(0) = 1$, thus $\lim_{x\to 0} \ell(x) \neq \ell(0)$ and so ℓ is **not** continuous at $x = 0$.

Hence ℓ is not continuous on $\mathbb{R}$.

5. i) Prove, by verifying the definition, that $\cos x$ is continuous on $\mathbb{R}$.
   Hint Make use of $\cos(x + y) = \cos x \cos y - \sin x \sin y$, valid for all $x, y \in \mathbb{R}$.

   ii) Prove that $\tan x$ is continuous for all $x \neq \pi/2 + k\pi, k \in \mathbb{Z}$.

Solution i) Let $a \in \mathbb{R}$ be given. We know that $\cos x$ is continuous at a if, and only if, $\cos(x + a)$ is continuous at $x = 0$. Thus we need examine

$$\begin{aligned}
\lim_{x\to 0} \cos(x + a) &= \lim_{x\to 0} (\cos x \cos a - \sin x \sin a) \\
&\qquad \text{by the assumption in the question,} \\
&= \left(\lim_{x\to 0} \cos x\right) \cos a - \left(\lim_{x\to 0} \sin x\right) \sin a \\
&\qquad \text{by the Product and Sum Rules for limits,} \\
&= 1 \times \cos a - 0 \times \sin a \\
&= \cos a = \cos(0 + a).
\end{aligned}$$

Thus $\cos(x + a)$ is continuous at $x = 0$ and hence $\cos x$ is continuous at a. True for all $a \in \mathbb{R}$ means cos is continuous on $\mathbb{R}$.

ii) Let $a \neq \pi/2 + k\pi$ for any $k \in \mathbb{Z}$ be given. Then

$$\lim_{x\to a} \tan x = \lim_{x\to a} \frac{\sin x}{\cos x} = \frac{\lim_{x\to a} \sin x}{\lim_{x\to a} \cos x}$$

by the Limit Law for Quotients. This is allowable since both limits exist (because sin and cos are everywhere continuous) and further $\lim_{x\to a} \cos x = \cos a \neq 0$ since $a \neq \pi/2 + k\pi$ for any $k \in \mathbb{Z}$. Thus

$$\lim_{x\to a} \tan x = \frac{\lim_{x\to a} \sin x}{\lim_{x\to a} \cos x} = \frac{\sin a}{\cos a} = \tan a.$$

Since the *limit* of tan at a equals the *value* of tan at a we have verified the definition that tan is continuous at a. Yet a was arbitrary subject to being not of the form $\pi/2 + k\pi$ for any $k \in \mathbb{Z}$, therefore tan is continuous for all $x \neq \pi/2 + k\pi$ for any $k \in \mathbb{Z}$.

6. Show that the hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$ are continuous on $\mathbb{R}$.

Solution Recall that

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

We know that e^x is continuous on $\mathbb{R}$ as is thus e^{-x}, either by the Quotient Rule since $e^{-x} = 1/e^x$ and $e^x \neq 0$ or by the Composition Rule $x \mapsto -x \mapsto e^{-x}$. Thus $\sinh x$ and $\cosh x$ are continuous on $\mathbb{R}$ by the Sum Rule.

For $\tanh x$ we use the Quotient Rule observing that $e^x + e^{-x}$ is never zero.

Composite Rule

7. i) State the Composite Rule for functions.

Evaluate

$$\lim_{x\to 0} \exp\left(\frac{\sin x}{x}\right).$$

ii) State the Composite Rule for continuous functions.

Prove that

$$\left|\frac{x+2}{x^2+1}\right|$$

is continuous on $\mathbb{R}$.

Solution Composite Rule for functions. *Assume that g is defined on a deleted neighbourhood of $a \in \mathbb{R}$ and $\lim_{x\to a} g(x) = L$ exists. Assume that f is defined on a neighbourhood of L and is continuous there. Then*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right). \tag{2}$$

i) Let

$$g(x) = \frac{\sin x}{x} \quad \text{and} \quad f(x) = \exp(x) = e^x.$$

Then g is defined on $\mathbb{R} \setminus \{0\}$ and $\lim_{x\to 0} g(x)$ exists, with value 1. Further f is defined on **all** of $\mathbb{R}$ and is continuous at $1 = \lim_{x\to 0} g(x)$. Thus we can apply the Composite Rule for functions to say

$$\begin{aligned} \lim_{x\to 0} \exp\left(\frac{\sin x}{x}\right) &= \lim_{x\to 0} f(g(x)) = f\left(\lim_{x\to 0} g(x)\right) \\ &= \exp\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = \exp(1) \\ &= e. \end{aligned}$$

ii) **Composite Rule for Continuous functions**. Assume that g is defined on a neighbourhood of $a \in \mathbb{R}$ and is continuous there and assume that f is defined on a neighbourhood of $g(a)$ and is continuous there, then $f \circ g$ is continuous at a.

Let

$$g(x) = \frac{x+2}{x^2+1} \quad \text{and} \quad f(x) = |x|.$$

We have seen in Questions 4i and 2 on that both g and f are continuous on all of $\mathbb{R}$. Hence by the Composite Rule for continuous functions we deduce that

$$f(g(x)) = \left|\frac{x+2}{x^2+1}\right|$$

is continuous at every $a \in \mathbb{R}$, i.e. is continuous on $\mathbb{R}$.